Algebraic Geometry of Twistor Spaces

N. Honda (Tokyo Tech)

Conference on Differential Geometry (LeBrun Fest.)

July 5, 2016

§ Anti-Self-Dual (ASD) metrics and Twistor Spaces

- (M^4, g) oriented Riem. 4-mfd

$$\rightsquigarrow \quad \Lambda^2 = \underset{SD}{\Lambda_+} \oplus \underset{ASD}{\Lambda_-}$$

- A connection ∇ on $E \xrightarrow{v.b.} M$ is ASD $:\Longleftrightarrow$ $\overset{\substack{\Lambda_+ \otimes \mathrm{End}\, E \\ \in}}{R_+} \equiv 0$

These attain the absolute infimum of the Yang-Mills functional.

- g is ASD $:\Longleftrightarrow$ $W_+ \equiv 0$ (conf. inv. condition)

$$Rm(g) \in \Lambda^2 \otimes \Lambda^2 = \begin{matrix} \Lambda_+ \\ \oplus \\ \Lambda_- \end{matrix} \otimes \begin{matrix} \Lambda_+ \\ \oplus \\ \Lambda_- \end{matrix}$$

(SD part of LC conn. on Λ_+)

//

W_+ : traceless part of $\underline{\Lambda_+ \otimes \Lambda_+ \text{ component}}$

These attain the absolute infimum of the Weyl functional

$$[g] \longmapsto \int_M |W|^2 \, dv_g .$$

$$\begin{tikzcd}
\pi^{-1}(p) \arrow[r, equal] & \ell_p \arrow[d] \arrow[r, phantom, "\subset"] & Z \arrow[d, "\pi"'] \arrow[r, phantom, "\subset"] & \Lambda_+ \arrow[dr, leftrightarrow] & \\
& p \arrow[r, phantom, "\in"] & M & & \mathfrak{sp}(1) = \operatorname{Im}\mathbb{H}
\end{tikzcd}$$

unit sphere bdle ($\operatorname{rk}\Lambda_+ = 3$)

- ℓ_p : the space of orthogonal cx str.-s on T_pM that are compatible with the orientation

$\rightsquigarrow$ Z has a natural almost cx str. J

Thm (Penrose, Atiyah-Hitchin-Singer)

J : integrable $\Longleftrightarrow$ g : ASD

The complex manifold Z is called the Twistor Space of an ASD mfd $(M^4, [g])$

Basic properties of twistor spaces

- $\ell_p = \pi^{-1}(p)$ is a cx submfd of Z, satisfying $N_{\ell_p/Z} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$

twistor line

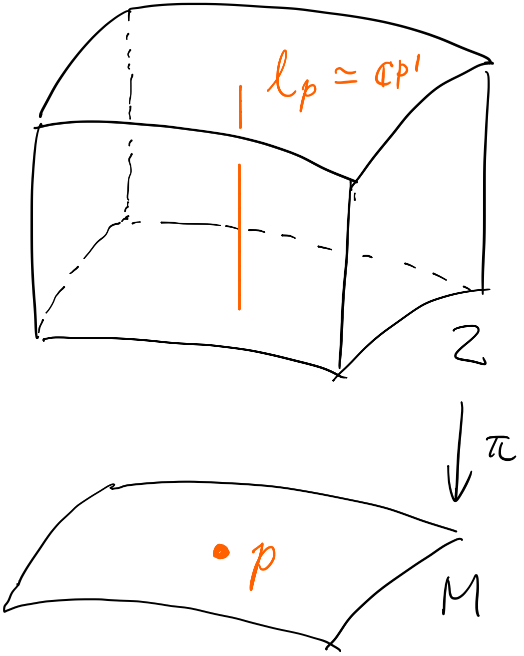
- $\exists \sigma : Z \longrightarrow Z$ anti-hol. involution s.t. $\sigma|_{\ell_p} \sim$ anti-podal map

real str.

Conversely, these structures define an ASD conformal structure on M.

$$\{\text{ASD str. on } M\} \overset{1:1}{\longleftrightarrow} \{\text{Twistor spaces over } M\}$$

Penrose corresp.



More basic properties of twistor spaces

- $K_Z|_\ell \simeq \mathcal{O}(-4)$ $\because$) By adjunction, $\underset{\mathcal{O}(-2)}{\underset{\wr\|}{K_\ell}} \simeq K_Z|_\ell \otimes \underset{\mathcal{O}(2)}{\underset{\wr\|}{\det N_{\ell/Z}}}$ //
- So $\kappa(Z) = -\infty$ if Z: cpt.
- $-K_Z|_\ell \simeq \mathcal{O}(4)$
- $-K_Z$ admits a natural square root F as a hol. line bdle.
- F: fundamental line bdle. This satisfies

$$F|_\ell \simeq \mathcal{O}(2), \quad \sigma^* F \simeq \overline{F}, \quad F \otimes F \simeq K_Z^{-1}$$

§ Two basic theorems on cpt twistor spaces

Thm (Hitchin '81) Z : cpt twistor space, $\exists$ Kähler metric on Z

$$\begin{tikzcd} \Longrightarrow \quad Z \simeq \mathbb{CP}^3 \arrow[d] & \text{or} & \mathbb{F} = \{(x, \ell) \mid x \in \ell \subset \mathbb{CP}^2\} \arrow[d] \\ S^4 & & \overline{\mathbb{CP}}^2 \end{tikzcd}$$

Outline of a proof. First notice that $K^{-1} > 0$ from Kählerity. So Z is Fano. Then investigate $|F|$ by using Riem. Roch, and Kodaira vanishing, with effective use of σ and ℓ. //

Thm (Campana '91) If Z is Moishezon (or of Fujiki Class $\mathcal{C}$) then the base sp. M is homeo. to $n\overline{\mathbb{CP}^2}$, where $0\,\overline{\mathbb{CP}^2} = S^4$

- X : Moishezon $:\Leftrightarrow$ X : birational to a proj. alg. variety $\Longleftrightarrow$ $\underset{\text{(alg. dim.)}}{a(Z)} = 3$
- $X : \mathcal{C}$ $\Longleftrightarrow$ X : birational to a cpt Kähler mfd.

<u>Outline of a proof</u>. Consider the Chow scheme of Z, parameterizing twistor lines through a point. These lines cover Z from compactness of Chow scheme. This means simply connectedness of Z, and so is M. Then a topological argument using Riem. Roch means $b_2^+(M) = 0$, which implies $M \underset{\text{(homeo)}}{\simeq} n\overline{\mathbb{CP}}^2$ by Freedman & Donaldson. //

Basic properties of a twistor space Z on $n\overline{\mathbb{CP}}^2$

- $a(Z) = \kappa(Z,F)$ $(= \kappa^{-1}(Z))$ (Poon, LeBrun)

- So Z : Moishezon $\Longleftrightarrow$ $\kappa^{-1}(Z) = 3$ (i.e. K^{-1} : big)

- $F^3 = 2(4-n)$

$n<4$	$n=4$	$n>4$
$+$	0	$-$

F^3

- $$\begin{array}{l} \chi(mF) \overset{R.R.}{=} \frac{1}{2}(4-n)m^3 + O(m^2) \\ \ \ \wedge\!| \\ h^0(mF) + h^2(mF) \\ \qquad\qquad\quad \| \ \text{by Hitchin vanishing thm if } Scal(g)>0 \\ \qquad\qquad\quad 0 \end{array}$$

- So $\kappa^{-1}(Z) = 3$ if $n<4$ and $Scal(g)>0$.

- If $n \geq 4$, $a(Z) < 3$ in general. But there are many Z on $n\overline{\mathbb{CP}}^2$, $n \geq 4$, which satisfy $a(Z) = 3$. These satisfy $F^3 < 0$ & F : big.

Classification of twistor spaces on $2\overline{\mathbb{CP}^2}$

Thm (Poon '86) If g is an ASD metric on $2\overline{\mathbb{CP}^2}$ satisfying $Scal(g)>0$, its twistor space Z has the following structure: $h^0(F)=6$, $Bs|F|=\phi$,

$$\begin{tikzcd} \Phi : Z \arrow[r, "|F|"] \arrow[dr, "birat."'] & \mathbb{CP}^5 \\ & X \arrow[u, phantom, "\cup\ (2,2)"] \end{tikzcd}$$

The moduli space is 1-dim, connected. $\square$

The first examples of Moishezon twistor spaces on $n\overline{\mathbb{CP}}^2$, n : arbitrary

Thm (LeBrun '91) There exists a family of ASD metrics on $n\overline{\mathbb{CP}}^2$ (explicitly constructible), whose twistor spaces have the following structure : $h^0(F) = 4$, (if $n > 2$) $\quad \mathrm{Bs}|F| = \underset{\mathbb{CP}^1}{\underset{\cong}{C}} \sqcup \underset{\mathbb{CP}^1}{\underset{\cong}{\overline{C}}}$

$$\begin{tikzcd}
Z \arrow[r, dashed, "|F|"] \arrow[rd, dashed, "\Phi"] & \mathbb{CP}^3 \\
 & Q \arrow[u, phantom, "\overset{(2)}{\cup}"] \\
\tilde{Z} \arrow[uu, "\text{blow up at } C \sqcup \overline{C}"] \arrow[ru, "\text{conic bdle}"'] &
\end{tikzcd}$$

discriminant locus

$$Q \supset Q \cap (\underbrace{H_1 \cup \cdots \cup H_n}_{\text{hyperplanes}}) \quad \text{hyperplanes} \Longrightarrow \text{moduli}$$

$$\begin{tikzcd}
\mathbb{C}^* \arrow[r, phantom, "\curvearrowright"] & Z \arrow[d] \\
S^1 \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\curvearrowright"] & n\overline{\mathbb{CP}}^2
\end{tikzcd}$$

These metrics admit an S^1-action, and the map Φ can be regarded as a quotient map of the $\mathbb{C}^*$-action. □

These twistor spaces are generalized to much more general S^1-actions :

Thm (H. 2009, 2010) For "many" S^1-action on $n\overline{\mathbb{CP}^2}$, we can construct a family of Moishezon twistor spaces having the following structure :

$$\begin{tikzcd}
Z \arrow[r, dashed, "|mF|"] \arrow[rd, dashed, "\Phi"] & \mathbb{CP}^N & & \\
 & \mathcal{T} \arrow[u, phantom, "\cup"] & \supset & \overset{\text{discriminant locus}}{\mathcal{T} \cap (\underbrace{H_1 \cup \cdots \cup H_k}_{\text{hyperplanes} \;\Longrightarrow\; \text{moduli}})} \\
\tilde{Z} \arrow[uu] \arrow[ru, "\text{conic bundle}"'] & & &
\end{tikzcd}$$

$$\boxed{\begin{tikzcd}
\mathbb{C}^* \arrow[r, phantom, "\curvearrowright"] \arrow[d, phantom, "\cup"] & Z \arrow[d] \\
S^1 \arrow[r, phantom, "\curvearrowright"] & n\overline{\mathbb{CP}^2}
\end{tikzcd}}$$

Φ can be regarded as a quotient map of the $\mathbb{C}^*$-action $\square$

Remarks on the theorem

- These twistor spaces are obtained as an S^1-equivariant deformations of the twistor spaces of Joyce metrics (1995), for suitable subgroups $S^1 \subset T^2$ ($\curvearrowright (n\overline{\mathbb{CP}^2}, \text{Joyce})$)

- The quotient surface $\mathcal{T}$ corresponds to Einstein-Weyl structure on 3-dim. space, and are (new) examples of minitwistors.

- $\mathcal{T}$ itself has a moduli.

Classification of twistor spaces on $3\overline{\mathbb{C}P}^2$

Thm (Poon, Kreußler-Kurke '92) If Z is a twistor space of an ASD metric on $3\overline{\mathbb{C}P}^2$ with Scal > 0, the twistor space satisfies $h^0(F) = 4$.

(1) If $\mathrm{Bs}|F| \neq \phi$, Z is a LeBrun twistor space.

(2) If $\mathrm{Bs}|F| = \phi$, Z has the following structure :

$$Z \xrightarrow[\left[\substack{\text{gen.}\\ 2\,:\,1}\right]]{|F|} \mathbb{C}P^3$$

$\cup$ branch div.

B : (singular) quartic

"double covering type"

Note : A substantial part of their analysis is to determine an explicit form of defining equation of B.

Classification of Moishezon twistor spaces on $4\overline{\mathbb{C}P}^2$

Thm (H.2014) If Z is a Moishezon twistor space on $4\overline{\mathbb{C}P}^2$, the anti-canonical map $\Phi : Z \xrightarrow{|2F|} \mathbb{C}P^N$ satisfies one of the following :

{Joyce} $\subset$

(1) Φ is birational over the image. ($N = 6$ or 8)

(2) Φ is $2:1$ over the image. $N = 4$.

scroll of planes

$$\begin{tikzcd} Z \arrow[r, dashed, "\Phi", "2:1"'] & p^{-1}(\Lambda) \arrow[d] & \subset & \mathbb{C}P^4 \arrow[d, "p : \text{projection}"] \\ & \Lambda & \subset & \mathbb{C}P^2 \end{tikzcd}$$

conic

The branch divisor is of the form $p^{-1}(\Lambda) \cap B$, where B is a quartic hypersurface.

(3) $\dim \Phi(Z) = 2$ ($N = 4, 5$ or 8), and Z is birational to a conic bundle over $\Phi(Z)$.

$\cup$

{LeBrun, generalized LeBrun}

Remarks on the theorem

- Defining eq. of $\Phi(2)$ can be given in a concrete form for any case.

  For example, in (1) (= birational type), if $N = (h^0(-K_Z) - 1 =)\ 6$,

  $\Phi(2) = (2, 2, 2)$ of some special kind.

- The case (2) can be regarded as a generalization of

  the similar one on $3\overline{\mathbb{C}P}^2$ to $4\overline{\mathbb{C}P}^2$.

  (Both have a double covering str, branch is determined by a quartic)

- A substantial part is devoted to determine an equation of B.

- Φ always has indeterminacy locus, and a concrete elimination can

  be given. ($\Rightarrow$ "birational geometry of twistor spaces")

Moishezon twistor spaces on $n\overline{\mathbb{CP}}^2$, $n > 4$

- No complete classification is obtained yet.
- If $h^0(F) \geq 4$, Z = LeBrun
- If $h^0(F) = 3$, $Z \simeq$ Campana - Kreußler (1998)

  ↑ having conic bdle str by $|F|$

- If $h^0(F) = 2$, known examples are :
  - generalized LeBrun (H. 2010). having $\mathbb{C}^*$-action
  - double covering type → next slide
- No example is known satisfying $h^0(F) \leq 1$.

with $\mathbb{C}^*$-action ↓ no $\mathbb{C}^*$-action ↓

Thm (H. 2008, 2015) For any $n > 4$, there exist families of twistor spaces having the following structure:

$$\begin{tikzcd}
Z \arrow[r, dashed, "|(n-2)F|"] \arrow[ddr, dashed, "\Phi", "2:1"'] & \mathbb{CP}^n \arrow[r, dashed, "p", "\text{lin. proj.}"'] & \mathbb{CP}^{n-2} \\
& \cup & \cup \\
& p^{-1}(\Lambda) \arrow[r] & \Lambda \simeq \mathbb{CP}^1
\end{tikzcd}$$

scroll of planes rat. norm. curve

branch divisor of $\Phi = p^{-1}(\Lambda) \cap B$, B : quartic hypersurface □

- These are characterized by presence of a member of $|F|$ whose pluri-anti-canonical system enjoies a double covering property.

On a proof of the theorem

- One difficulty is to show that the double covering map from the member extends to Z (to give $\Phi : Z \underset{2:1}{\dashrightarrow} p^{-1}(\Lambda)$).

- Another difficulty is to derive a constraint for the branch divisor. (Defining equation of the quartic hypersurface B.)

- In these analysis, interesting birational geometry arises.

Further directions

- To extend the last theorem in full generality.

  (It has turned out, there exist a large number of families of Z of double covering type, which contain the above two families as very special cases.)

- To show that

  generalized LeBrun & double covering type

  exhaust Z satisfying $h^0(F) = 2$.

- To pursue a connection with Fano 3-folds

- Recall $a(Z) = 3 \Leftrightarrow |K_Z^{-1}|$: big $\Leftrightarrow \kappa^{-1}(Z) = 3$

- Structure of surfaces with $\kappa^{-1}(X) = 2$ is as follows :

Thm (Sakai '84) The anti-canonical model of X has only isolated $\mathbb{Q}$-Gorenstein singularities, and mK^{-1} is ample for some $m \in \mathbb{N}$. □

- Analogous result for 3-fold seems to be not known.

(Nef property is not assumed.)

Projective models of Moishezon twistor spaces

$$2\,\overline{\mathbb{C}P}^2 \qquad Z \xrightarrow[\text{bir}]{|F|} X_{2,2} \subset \mathbb{C}P^5$$

quartic.

$$3\,\overline{\mathbb{C}P}^2 \qquad Z \xrightarrow[2:1]{|F|} \mathbb{C}P^3 \supset B$$

$$4\,\overline{\mathbb{C}P}^2 \qquad Z \xrightarrow[\text{bir.}]{|2F|} X_{2,2,2} \subset \mathbb{C}P^6$$

$$Z \xrightarrow[\text{bir.}]{|2F|} X \subset \mathbb{C}P^8$$

intersection of 10 quadrics

All these are limits of Fano 3-folds of the same types.

So perhaps all Moishezon Z are birational to Fano 3-folds with singularities.